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Khovanov homotopy types and the Dold-Thom functor

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Abstract. We show that the spectrum constructed by Everitt and Turner as a possible Khovanov homotopy type is a product of Eilenberg-MacLane spaces and is thus determined by Khovanov homology. By using the Dold-Thom functor it can therefore be obtained from the Khovanov homotopy type constructed by Lipshitz and Sarkar.

A *Khovanov homotopy type* is a way of associating a (stable) space to each link L so that the classical invariants of the space yield the Khovanov homology of L. There are two recent constructions of Khovanov homotopy types, using different techniques and giving different results [3,6]. In [3] homotopy limits were employed to build an Ω-spectrum $\mathbf{X}_{\bullet}L = \{X_k(L)\}$ with the following properties:

(i). the homotopy type is a link invariant, and
(ii). the homotopy groups are Khovanov homology:

$$\pi_i(\mathbf{X}_{\bullet}(L)) = Kh^{-i}(L).$$

The main goal of this note is to prove the following result.

Theorem 1. *Each of the spaces $X_k(L)$ is homotopy equivalent to a product of Eilenberg-MacLane spaces.*

In [6] the programme of Cohen, Jones and Segal [2] was generalized to produce a suspension spectrum $\mathcal{X}_{Kh}(L)$ with the following properties:

(i). the homotopy type is a link invariant, and
(ii). the reduced cohomology is Khovanov homology:

$$\widetilde{H}^i(\mathcal{X}_{Kh}(L)) = Kh^i(L).$$

As a corollary we obtain that $\mathbf{X}_{\bullet}(L)$ is homotopy equivalent to the infinite symmetric product of $\mathcal{X}_{Kh}(L)$.

To prove Theorem 1 we use the explicit model, due to McCord [8], of the Eilenberg-MacLane spaces. Given a monoid G and a based topological space X, let $B(G,X)$ denote the set of maps

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$u\colon X \to G$ such that $u(x) = 0$ for all but finitely many $x \in X$. Then $B(G,X)$ is a monoid, and if G is a group (the case of interest) then $B(G,X)$ is a group. Moreover, when G is an abelian topological group the set $B(G,X)$ can be topologized in a natural way so that the group operation is continuous. This construction has nice functoriality: letting $\mathsf{Ab}, \mathsf{Top}_*$ and AbTop denote respectively the categories of abelian groups, based topological spaces and topological abelian groups, one has the following result.

Proposition 1. [8, Proposition 6.7] *McCord's construction is a bifunctor*

$$B(-,-)\colon \mathsf{Ab} \times \mathsf{Top}_* \to \mathsf{AbTop}.$$

Furthermore, as special case of [8, Theorem 11.4], for an abelian group G the space $B(G,S^n)$ is the Eilenberg-MacLane space $K(G,n)$. Thus we may take as *the* Eilenberg-MacLane space functor:

$$B(-,S^n)\colon \mathsf{Ab} \to \mathsf{AbTop}.$$

Conversely, the following is [4, Corollary 4K.7, p. 483] (apparently originally due to Moore; cf. [8, p. 295]):

Proposition 2. *A path-connected, commutative topological monoid is a product of Eilenberg-MacLane spaces.*

The spaces $X_k(L)$ are built as homotopy limits of diagrams of spaces. Recall that given a small category C and a (covariant) functor $D\colon \mathsf{C} \to \mathsf{Top}_*$ (a diagram), that $\operatorname{holim}_{\mathsf{C}} D$ is constructed as follows (see, e.g., [1, Section 11.5] or the concise notes [9, Section 3.7]). Consider the product

$$\prod_{\sigma\in N(\mathsf{C})} \mathrm{Hom}(\Delta^n, D(c_n)) = \prod_{n\geq 0} \prod_{\substack{c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} c_n \\ \alpha_i \neq \mathrm{Id}}} \mathrm{Hom}(\Delta^n, D(c_n)) \tag{1}$$

where $N(\mathsf{C})$ is the subset of the nerve of C consisting of all sequences of composable morphisms $\sigma = (c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} c_n)$ in which none of the morphisms are identity maps, and Hom denotes the space of continuous maps from the standard n-simplex. The homotopy limit $\operatorname{holim}_{\mathsf{C}} D$ is the subspace of this product consisting of those tuples $(f_\sigma)_{\sigma\in N(\mathsf{C})}$ such that the following diagrams commute:

$$\begin{tikzcd} \Delta^{n-1} \arrow[r, "f_{d_i\sigma}"] \arrow[d, "d^i"'] & D(c_n) \arrow[d, "\mathrm{Id}"] \\ \Delta^n \arrow[r, "f_\sigma"] & D(c_n) \end{tikzcd} \tag{2}$$

for each $0 < i < n$, and

$$\begin{tikzcd} \Delta^{n-1} \arrow[r, "f_{d_0\sigma}"] \arrow[d, "d^0"'] & D(c_n) \arrow[d, "\mathrm{Id}"] \\ \Delta^n \arrow[r, "f_\sigma"] & D(c_n) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Delta^{n-1} \arrow[r, "f_{d_n\sigma}"] \arrow[d, "d^n"'] & D(c_{n-1}) \arrow[d, "D(\alpha_n)"] \\ \Delta^n \arrow[r, "f_\sigma"] & D(c_n) \end{tikzcd} \tag{3}$$

corresponding to the cases $i = 0$ and $i = n$, respectively. Here the map d^i denotes the i^{th} face inclusion, $d_i\sigma = (c_0 \xrightarrow{\alpha_1} \cdots c_{i-1} \xrightarrow{\alpha_{i+1}\alpha_i} c_{i+1} \cdots \xrightarrow{\alpha_n} c_n)$ when $0 < i < n$, and d_0, d_n similarly.

The following is well-known, but for completeness we give its (short) proof.

Proposition 3. *Let $D\colon \mathsf{C} \to \mathsf{Top}_*$ be a diagram of topological abelian groups and continuous group homomorphisms. Then the homotopy limit of D is a topological abelian group.*

Proof. Pointwise addition makes the set $\mathrm{Hom}(\Delta^n, D(c_n))$ into an abelian group, and the product in formula (1) is the product (topological abelian) group. It remains to see that the diagrams (2) and (3) describe a subgroup of this product. Suppose that tuples (f_σ) and (g_σ) make these diagrams commute. Then the first two diagrams automatically commute for the pointwise sum $(f_\sigma + g_\sigma)$. The third diagram for the pointwise sum becomes,

$$\begin{tikzcd}
\Delta^{n-1} \arrow[r] \arrow[d, "d^n"'] & \Delta^{n-1} \times \Delta^{n-1} \arrow[r, "f_{d_n\sigma} \times g_{d_n\sigma}"] \arrow[d, "d^n \times d^n"] & D(c_{n-1}) \times D(c_{n-1}) \arrow[r, "+"] \arrow[d, "D(\alpha_n)\times D(\alpha_n)"] & D(c_{n-1}) \arrow[d, "D(\alpha_n)"] \\
\Delta^n \arrow[r] & \Delta^n \times \Delta^n \arrow[r, "f_\sigma \times g_\sigma"] & D(c_n) \times D(c_n) \arrow[r, "+"] & D(c_n)
\end{tikzcd}$$

for which the first square obviously commutes, the second commutes since f and g are in the prescribed subspace and the third commutes from the fact that $D(\alpha_n)$ is a group homomorphism. The inverse operation is similarly seen to be closed, hence the subspace defined above is a subgroup. □

Proof (Proof of Theorem 1). Let L be an oriented link diagram with c negative crossings. The space $X_k(L)$ is constructed as follows. Let I denote the category with objects $\{0,1\}$ and a single morphism from 0 to 1, and I^n the product of I with itself n times. Let $\overline{0}$ be the initial object in I^n, and let $\mathbf{P}$ be the result of adjoining one more object to I^n and a single morphism from the new object to every object except $\overline{0}$.

In [3] it is shown that there is a functor $F\colon \mathbf{P} \to \mathsf{Ab}$ such that the i^{th} derived functor of the inverse limit, $\varprojlim_{\mathbf{P}}{}^i F$, is isomorphic to the i^{th} unreduced Khovanov homology of L. The space $X_k(L)$ is constructed by composing this functor with the Eilenberg-MacLane space functor $K(-,k+c)$ and taking the homotopy limit of the resulting diagram of spaces.

We may now use the explicit model for Eilenberg-MacLane spaces given by McCord. By applying Proposition 1 we define a diagram $D\colon \mathbf{P} \to \mathsf{AbTop}$ as the composition

$$\mathbf{P} \xrightarrow{F} \mathsf{Ab} \xrightarrow{B(-,S^{k+c})} \mathsf{AbTop}.$$

By the homotopy invariance property of the homotopy limit construction we have

$$X_k(L) \simeq \mathrm{holim}_{\mathbf{P}} D.$$

By Proposition 3, the homotopy limit on the right is itself a topological abelian group, and hence, by Proposition 2, a product of Eilenberg-MacLane spaces. □

Corollary 1. *The homotopy type of $\mathbf{X}_\bullet(L)$ is determined by $Kh(L)$.*

The spectrum $\mathcal{X}_{Kh}(L) = \{\mathcal{X}_{Kh}^{(k)}(L)\}$ constructed in [6] has the additional property that the cellular cochain complex of the space $\mathcal{X}_{Kh}^{(k)}(L)$ is isomorphic to the Khovanov complex of L (up to shift). It follows from the description of the Khovanov homology of the mirror image (see [5]) that

$$\widetilde{H}_i(\mathcal{X}_{Kh}(L)) = Kh^{-i}(-L)$$

where $-L$ denotes the mirror of L. The infinite symmetric product $\mathrm{Sym}^\infty \mathcal{X}^{(k)}_{Kh}(L)$ is seen from the Dold-Thom theorem to be

$$\mathrm{Sym}^\infty \mathcal{X}^{(k)}_{Kh}(L) = \prod_n K(\widetilde{H}_n(\mathcal{X}^{(k)}_{Kh}(L)), n)$$

from which we have the following.

Corollary 2. *For large enough k, the space $X_k(-L)$ is homotopy equivalent to the infinite symmetric product* $\mathrm{Sym}^\infty \mathcal{X}^{(k)}_{Kh}(L)$.

We end by noting that the analogue or Theorem 1 for the spectra $\mathcal{X}_{Kh}(L)$ is not true. For all alternating knots $\mathcal{X}_{Kh}(L)$ is a wedge of Moore spaces [6], however there are examples of non-alternating knots for which $\mathcal{X}_{Kh}(L)$ is not a wedge of Moore spaces (see [7]).

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